parameter. It shows that pit density increases with the burning time in a low burning pressure region, but in a high burning pressure region it increases to a maximum value and then decreases due to the agglomeration of neighboring pits (or convergence) with time. Regarding the effect of burning pressure, the pit density increases enormously with pressure during the initial burning time ($0 \sim 0.4$ sec). In the second stage, however, where burning time is longer (0.4 sec$\sim$), the high burning pressure leads to the high rate of convergence of pits, so that the pit density in a high pressure region becomes lower than that in a low burning pressure region.

In regard to pit formation, there is a critical pressure of about 25kg/cm^2 which is not affected by acceleration level, burning time, or the propellants used for this investigation. Although no pits exist on extinguished combustion surfaces below the critical pressure, many agglomerated spheres are observed on them. Therefore, it is necessary to consider another factor for pitting of accelerated combustion surface in addition to the agglomeration of retained aluminum particles. It has been reported that there is a critical relationship between pit density and acceleration level, and that the agglomerated aluminum spheres cannot promote the burning rate of the propellant under a critical value of acceleration.[5,6] Thus, it seems that there is an analogous mechanism to the burning pressure, and the phenomenon will be explained through the lift distance increase of the spheres by decreasing pressure, which causes the reduction of the amount of heat feedback to combustion surface.

Many discrete pits with sharp bottoms also can be seen on the extinguished combustion surfaces of PB04 nonaluminized propellant, which are made by AP particles contained in the propellant. The PB04 propellant has a lower pit density in the low acceleration field, and a higher pit density in the high acceleration field in comparison with the aluminized propellant PB05. But this high pit density by AP particles does not always yield high acceleration sensitivity of the burning rate of aluminized propellants. That is, the result of the PB10 propellant containing 2.0% aluminum shows that the extinguished combustion surfaces are covered by continuous round-bottomed pits which are characteristic of pits formed by agglomerated aluminum spheres. Thus, it is concluded that aluminum particles have a dominant role in the pitting at the initial burning time, and the pitting by AP particles need not be considered for high-aluminized composite propellants.

Conclusions

Extinction tests of slab motors were conducted to investigate the pitting on the accelerated combustion surface responsible for the acceleration-induced burning rate increase. A pit density chart on the burning pressure and the burning time with acceleration as a parameter was obtained for an aluminized propellant. It shows that pit density increases with the acceleration and the burning pressure initially, but that it decreases with burning time after reaching a maximum value in high burning pressure region. This decrease of pit density is due to the convergence of neighboring pits, which can be distinguished by the shape of continuous shallow pits on the extinguished combustion surfaces. The initial strong pressure dependence can be explained by formulating the accumulation rate of retained aluminum particles on the accelerated combustion surface. For all propellants used, there was a critical burning pressure of 25kg/cm^2, and no pits were observed on the combustion surfaces below this value. This critical pressure may be explained through the heat feedback mechanism by the agglomerated aluminum sphere. It was shown by using a nonaluminized propellant that AP particles also had the potential to form many pits on the accelerated combustion surfaces. These pits, however, need not be considered except in the case of non/low-aluminized propellants, because of the dominant effect of aluminum particles on pit formation.

References

[1] Niioka, T., Mitani, T., and Ishii, S., "Observation of the Combustion Surface by Extinction Tests of Spinning Solid Propellant Rocket Motors," *Proceedings of the 11th International Symposium of Space Technology and Science,* 1975, pp. 77-82.

[2] Northam, G. B., "Effects of the Acceleration Vector on Transient Burning Rate of an Aluminized Solid Propellant," *Journal of Spacecraft and Rockets,* Vol. 8, Nov. 1971, pp. 1133-1137.

[3] Niioka, T., Mitani, T., and Ishii, S., "Transient Period of the Acceleration-Produced Burning Rate Augmentation," *AIAA Journal,* Vol. 14, 1976, to be published.

[4] Doin, B., Brulard, J., and Larue, P., "Facilities for Studying the Combustion of Spinning Metallized Solid Propellants," TN-229, 1974, ONERA.

[5] Niioka, T. and Mitani, T., "Independent Region of Acceleration in Solid Propellant Combustion," *AIAA Journal,* Vol. 12, Dec. 1974, pp. 1759-1761.

[6] Mitani, T. and Niioka, T., "Firing Tests of Spinning 200ϕ-Solid Propellant Rocket Motors," *19th Symposium on Space Science and Technology,* Nagoya, Japan, 1975, pp. 201-204, in Japanese.

Technical Comments

Comment on "Application of Hamilton's Law of Varying Action"

Arthur E. Bryson Jr.
Stanford University, Stanford, Calif.

WE believe that the theory in Ref. 1 is basically incorrect. From Ref. (2) Hamilton's Principle for holomonic nonconservative systems may be stated as follows: the first variation of the time integral of system kinetic energy plus the time integral of system virtual work is zero, i.e.

$$\delta \int_{t_o}^{t_1} T(q_i, \dot{q}_i)dt + \int_{t_o}^{t_1} \sum_j Q_j (q_i, \dot{q}_i)\delta q_j \, dt = 0$$

where q_1, ---, q_n are the generalized coordinates, T is kinetic energy, Q_1, --- Q_n are generalized forces, and δ indicates an infinitesimal variation; $\sum_j Q_j \, \delta q_j$ is the virtual work, where the Q_j are defined by the fact that the work done on the system by the Q_j is given by the path-dependent line integral

$$W = \sum_j \int_{q_j(t_o)}^{q_j(t_1)} Q_j (q_i, \dot{q}_i) \, dq_j$$

Bailey claims that a variational principle for nonconservative systems is (cf. Eq. (4) of Ref. 1)

$$\delta \int_{t_o}^{t_1} (T+W)dt = \sum_j \frac{\partial T}{\partial \dot{q}_j} \delta q_j \mid t_o{}^{t}$$

but does not define the work W so that $\delta \int_{t_0}^{t_1} W dt$ is a meaningless expression. In his first example virtual work (instead of work) is used in Eq. (5). In his second example, in Eq. (11), work is incorrectly defined for the hinge moments of the double pendulum. The correct definitions are

Received Jan. 30, 1976.
Index category: Structural Dynamic Analysis.

*Professor and Chairman, Dept. of Aeronautics and Astronautics. Fellow AIAA.

$$W=\int_{q_1(o)}^{q_1(t)}(-c_1\dot{q}_1-k_1q_1+T_1)dq_1+\int_{q_2(0)}^{q_2(t)}(-c_2\dot{q}_2-k_2q_2+T)dq_2+\cdots$$

where the integrals are line integrals in the $\dot{q}_1$, q_1 phase space; thus the work done is path-dependent. The solution to the second example appears to be correct, indicating that Bailey again used virtual work, not work, in determing the solution.

We agree with Bailey that there is some confusion on this subject in the recent literature. For example, Goldstein[3] has an incorrect derivation of Lagrange's equations for nonconservative systems using a principle similar to that of Ref. 1 and Lanczos apparently chooses not even to discuss the issue in his treatise.[4] However, Whittaker in 1904 gave a complete discussion (see Ch. IX of Ref. 2).

References

[1]Bailey, C.D., "Application of Hamilton's Law of Varying Action," *AIAA Journal,* Vol. 13, Sept. 1975, pp. 1154-1157.

[2]Whittaker, E.T., *Analytical Dynamics,* Dover, N.Y., 1944 (4th Edition).

[3]Goldstein, H., *Classical Mechanics,* Addison-Wesley, 1950, pp. 38-39.

[4]Lanczos, C., *The Variational Principles of Mechanics,* Toronto, 1970 (4th Edition).

Reply by Author to A. E. Bryson Jr.

Cecil D. Bailey*
The Ohio State University, Columbus, Ohio

IN the last paragraph of his comment, Professor Bryson has made, in our humble opinion, the understatement of the century. He agrees with us that there is some confusion on this subject; but, he then restricts his comment to the recent literature. He points out Goldstein[1] as being incorrect; he mentions Lanczos[2]; he doesn't mention Osgood[3]; and finally, he bases his conclusions on Whittaker.[4] Who, pray tell, are we to believe? The confusion is not recent. It has been with us for a very long time and, without doubt, will continue for years to come. This was brought home to us by the statement of an internationally recognized authority who said, "I do not like energy methods because almost every writer has a different treatment." Progress in our work was delayed almost a year because we thought Goldstein to be correct. Now Professor Bryson believes that our theory is "basically incorrect."

Hamilton did not furnish a variational principle in the sense that we are taught variational principles in the variational calculus, and we make no claim of any variational principle. Hamilton furnished the Law of Varying Action for the motion of matter in time-space.[5,6] The authority for this is in Hamilton's own words[5]; but, nowhere have we found direct analytical solutions to time-dependent, nonconservative systems from energy considerations alone, which Hamilton implied could be achieved.[6]

Our results are correct because our theory is correct; but, a definitive paper on this subject is yet to be written. The reasons for this are set forth in Ref. 7 and Ref. 8. A copy of Ref. 8 is available from either the NASA Langley Research Center or from the author.

We wish to express our appreciation to Professor Bryson for writing. If our theory appears incorrect, it is because our concepts are different.[7,8] We define work in the same way that Professor Bryson finds it defined in Whittaker; and, if we should ever desire to calculate the work explicitly, we would calculate it as indicated by Professor Bryson. There is no inconsistency in its use as presented in our published papers[9-12] for reasons which we will present, "...at another time in another place."[13]

Received March 31, 1976; revision received May 3, 1976.
Index category: Structural Dynamic Analysis.
*Professor. Member AIAA.

References

[1]Goldstein, H., *Classical Mechanics,* Addison-Wesley, Reading, Mass. 1950.

[2]Lanczos, C., *The Variational Principles of Mechanics,* Toronto, 1970 (4th ed).

[3]Osgood, W. F., *Mechanics,* Macmillan, N.Y., 1937.

[4]Whittaker, E. T., *Analytical Dynamics,* Dover, N.Y., 1944.

[5]Hamilton, W.R., "On a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London,* Vol. 124, 1834, pp. 247-308.

[6]Hamilton, W. R., "Second Essay on a General Method in Dynamics," *Philosophical Transactions of the Royal Society of London,* Vol. 125, 1835, pp. 95-144.

[7]Bailey, C. D., "A New Look at Hamilton's Principle,"- *Foundations of Physics,* Vol. 5, No. 3, Sept. 1975, pp. 433-451.

[8]Bailey, C.D., "Application of the General Energy Equation: A Unified Approach to Mechanics," Final Report, NASA Grant NGR 36-008-197, Aug. 1975.

[9]Bailey, C. D., "The Method of Ritz Applied to the Equation of Hamilton," *Computer Methods in Applied Mechanics and Engineering,* Vol. 7, 1976, pp. 235-247.

[10]Bailey, C. D., "Application of Hamilton's Law of Varying Action," *AIAA Journal,* Vol. 13, Sept. 1975, pp. 1154-1157.

[11]Bailey, C.D., "Exact and Direct Analytical Solutions to Vibrating Systems with Discontinuities," *Journal of Sound and Vibration,* Vol. 44, Jan. 1976, pp. 15-25.

[12]Bailey, C. D., "Hamilton, Ritz and Elastodynamics," ASME-*Journal of Applied Mechanics,* Paper 76-APM-R.

[13]Smith, D. R. and Smith, D. V., Jr., "Reply by Authors to S. F. Felszeghy and C. D. Bailey," *AIAA Journal,* Vol. 13, Nov. 1975, p. 1540.

Comment on "Practical Aspect of the Generalized Inverse of a Matrix"

Bertrand T. Fang*
EG&G/Washington Analytical Services Center, Inc.
Wolf Research and Development Group,
Riverdale, Md.

Hassig[1] obviously did not read the extensive references he cited on generalized inverse. The uniqueness of the generalized inverse was stated in the very beginning of Penrose (Hassig's Ref. 1). The very first equations of Greville (Hassig's Ref. 2) give Hassig's main results, Eq. (11) and (12). It is also well-known that when the solution to Hassig's Eq. (1) is not unique, the generalized inverse gives the minimum norm solution. Hassig also misses the basic concept and usefulness of generalized inverse by unduly restricting himself to matrices of maximal rank ("linearly independent" in Hassig's somewhat misleading terminology). The article by Greville is meant to be an elementary exposition on generalized inverse and is easy to read. Also, most modern engineering tests on control and estimation theory have a brief treatise on the subject.

Reference

[1]Hassig, H. J., "Practical Aspect of the Generalized Inverse of a Matrix," *AIAA Journal,* Vol. 13, Nov. 1975, pp. 1530-1531.

Received Feb. 27, 1976.
Index Category: Aeroelasticity and Hydroelasticity.
*Senior Scientific Specialist.